Inverse avalanches in the Abelian sandpile model

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We define and study the inverse of particle addition process in the Abelian sandpile model. We show how to obtain the unique recurrent configuration corresponding to a single particle deletion by a sequence of operations called inverse avalanches. We study the probability distribution of s_1, the number of "untopplings" in the first inverse avalanche. For a square lattice, we determine Prob(s_1) exactly for $s_1=0$, 1, 2, and 3. For large s_1, we show that Prob(s_1) varies as $s_1^{-11/8}$. In the direct avalanches, this is related to the probability distribution of the number of sites which topple as often as the site where the particle was added. These results are verified by numerical simulations.

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The Abelian sandpile model (ASM) is a probabilistic cellular automaton model which has attracted a lot of attention in recent years. The model was proposed by Bak, Tang, and Wiesenfeld as a simple discrete model to illustrate the concept of self-organized criticality [1]. The most attractive feature of this model is that the operators corresponding to sand grain addition at different sites commute, and this property enables the analytical determination of many of the properties of this model [2]. These include the characterization of the critical steady state, the two point correlation function, and the spectrum of relaxation times. In the case when there is a preferred direction of particle transfer (to take into account the presence of an external field like gravity), the model is in the universality class of the well studied voter model. This case has been solved exactly, and it was found that the model has upper critical dimension 3, and all the critical exponents describing avalanche size distribution have been determined in all dimensions [3].

For the undirected case, the problem has been solved exactly only on the Bethe lattice [4]. The most often studied case is the two dimensional square lattice, both by simulations [5,6] and theoretically. In this case, using the general equivalence of ASM's to the $q \rightarrow 0$ limit of the q-state Potts model, and known values of exponents from conformal field theory, all the exponents can be expressed in terms of one unknown exponent [7]. Only recently has it been possible to calculate the fractional number of sites having different heights in the steady state in two dimensions [8]. It is also known that correlations between sites with minimum allowed height vary as r^{-4} in the critical state, where r is the separation between sites, in the bulk, and on the surface [9,10]. However, so far it has not been possible to relate any of the exponents referring to distribution of sizes of avalanches to known exponents. The present paper is an effort in this direction.

In an earlier paper we have shown that there exists a one to one correspondence between the recurrent configurations of ASM and the spanning tree configurations on the same lattice [7]. The equivalence between these problems implies that all critical exponents defined in the spanning tree problem would have corresponding exponents in the ASM case. For example, the fractal dimension of chemical paths on the spanning tree is related to the way the duration of avalanches scales with their linear size [7]. If from a spanning tree we delete a bond at random, the probability distribution of the number of sites disconnected is known to have power law tails. In two dimensions, this probability varies as $s^{-11/8}$, where s is the number of sites disconnected [11]. In this paper we show that the exponent in the ASM case that corresponds to deleting an edge in the spanning tree problem is related to the process of *removing* a particle in the sandpile. To get back a recurrent configuration, we have to perform a number of operations called inverse avalanches. We study the statistics of the size of the first inverse avalanche denoted by s_1 and show that it can be calculated exactly for small s_1. For large s_1, Prob(s_1) varies as $s_1^{-11/8}$. We show that the same exponent characterizes the distribution of the number of sites which topple as often as the site where the particle was added in a direct avalanche. These results are verified by numerical calculations.

We consider the ASM on a lattice with N sites, the toppling rules being given by an $N \times N$ integer matrix Δ. In the following, we shall assume that Δ is a symmetrical matrix. Let C be a recurrent configuration of the pile, and C_f denote the recurrent configuration obtained by adding a particle at site i to C and relaxing the pile. If a_i denotes the operator corresponding to particle addition at i, we get

$$C_f = a_i C \ . \tag{1}$$

We have shown earlier that the operators $\{a_i\}$, $i=1$ to N, form an Abelian group under multiplication [2]. In particular, over the space of recurrent configurations,

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operators a_i^{-1} are well defined operators.

Given a configuration C_f, we can obtain the configuration $C=a_i^{-1}C_f$ as follows: Let the height of the sandpile in C_f at site j be z_j. We write

$$C_f=\{z_j\}\ . \tag{2}$$

We decrease the height z_i by 1 and call the resulting configuration C':

$$C'=\{z_j-\delta_{ij}\}\ . \tag{3}$$

We can check if C' is recurrent by using the burning algorithm [2]. For ASM models where Δ is not a symmetrical matrix, the burning test is not sufficient to ensure recurrence [12]. Since we are restricting ourselves to symmetric Δ, we shall not discuss the modifications necessary in this algorithm to make it applicable for nonsymmetric Δ. If C' is recurrent, then by the uniqueness of the inverse, $C=C'$. If not, it must contain a forbidden subconfiguration (FSC). An FSC is an assignment of heights to a subset of all sites of the lattices, which cannot appear in the steady state of the ASM. For example, if there are two sites α and β such that $\Delta_{\alpha\beta}=\Delta_{\beta\alpha}=-1$, then it is easy to see that any configuration with $z_\alpha=z_\beta=1$ cannot appear in the steady state of the ASM, and $z_\alpha=z_\beta=1$ is an FSC. For a general definition, see Ref. [2].

Let the FSC in C' be denoted by F_1. Define "untoppling" at site j by the operation

$$z_k \rightarrow z_k+\Delta_{kj}\ . \tag{4}$$

Clearly, untoppling is an inverse of the toppling. We untopple once at each site in F_1 and call the resulting configuration C''. Now we check if C'' is recurrent. If so, then $C=C''$. If not, then it contains an FSC; call it F''. We untopple once at all sites in F''. Call the resulting configuration C'''. And then test of recurrence of C''', . . . , and so on, until a recurrent configuration is reached, and the process stops.

We call this process the inverse avalanche process, and untopplings at $F_1, F_2, \ldots$, etc., are called the first, second, etc., inverse avalanches. The number of untopplings in an inverse avalanche will be called its size. Thus the size of the first inverse avalanche is the number of sites in F_1. In the following, we shall represent it by s_1. In general, s_i is the size of the ith inverse avalanche.

It is easy to see that if adding a particle at i in C gives rise to n_j topplings at a site j before the stable configuration C_f is reached, and deleting a particle at i from C_f, the number of untopplings at j is $\bar{n}_j$, then we have

$$n_j=\bar{n}_j \tag{5}$$

for all j. In particular, if there are a total of s topplings in going from C to C_f, we have

$$s=s_1+s_2+s_3+\cdots\ . \tag{6}$$

It turns out that calculating the statistical properties of the random variable s_1 is much easier than calculating those of s. For concreteness, we restrict ourselves to a two dimensional square lattice in the following. We assume a lattice of size $L\times L$ with L large, and open boundary conditions. The heights at sites take values 1 to 4. If the height at any site exceeds, it topples, and the height decreases by 4, while the height at each nearest neighbor increases by 1.

Let $\mathrm{Prob}(s_1=r)$ denote the probability that, in the steady state of the ASM, removing a particle from a randomly selected site causes a first inverse avalanche with exactly r untopplings. Clearly, $\mathrm{Prob}(s_1=0)$ equals the probability that no toppling occurs when we add a particle at a random site i in the critical state 4 ASM. Thus we get

$$\mathrm{Prob}(s_1=0)=1-\mathrm{Prob}(z_i=4)\ . \tag{7}$$

The right-hand side has been calculated by Priezzhev [8].

Now consider the case $s_1=1$. This can occur only if $z_{if}=1$. Hence we get

$$\mathrm{Prob}(s_1=1)=\mathrm{Prob}(z_{if}=1) = \frac{2}{\pi}\left[1-\frac{2}{\pi}\right]\approx 0.073\,636\ , \tag{8.1}$$

using the known result of height probabilities [8,9].

Calculating $\mathrm{Prob}(s_1=2)$ is not much harder. In this case F_1 must have only two sites, and the only possibility is two adjacent sites with height 1 each. Hence we must have $z_{if}=2$, with one of the four neighbors of i having height 1. We thus get

$$\mathrm{Prob}\,(S_1=2) = 4\,\mathrm{Prob}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}\right) \approx 0.041324 \tag{8.2}$$

where we have used obvious notation for the probability that, in the self-organized critical state, the preselected cluster of sites has the height subconfiguration shown, and used the earlier calculated results of Ref. [9].

For any finite value of s_1, there are only a denumerable number of FSC's, and the probabilities of each subconfiguration can be calculated as determinants of finite matrices using the method described in [9]. Thus we get

$$\mathrm{Prob}\,(S_1=3) = 6\,\mathrm{Prob}\left(\begin{array}{ccc} 2 & 2 & 1 \end{array}\right) + 12\,\mathrm{Prob}\left(\begin{array}{cc} 2 & 1 \\ 2 & \end{array}\right) \tag{8.3}$$

$$\mathrm{Prob}\,(S_1=4) = 8\,\mathrm{Prob}\left(\begin{array}{cccc} 2 & 2 & 2 & 1 \end{array}\right) + 32\,\mathrm{Prob}\left(\begin{array}{ccc} 2 & 2 & 1 \\ 2 & & \end{array}\right) + 16\,\mathrm{Prob}\left(\begin{array}{ccc} & 2 & 1 \\ 2 & 2 & \end{array}\right)$$
$$+ 16\,\mathrm{Prob}\left(\begin{array}{ccc} & 1 & \\ 2 & 3 & 1 \end{array}\right) + 16\,\mathrm{Prob}\left(\begin{array}{cc} 3 & 1 \\ 2 & 2 \end{array}\right)\ . \tag{8.4}$$

We note that a similar series is encountered in calculating the probability that on deleting a bond at random in a spanning tree, exactly s_c sites get disconnected [11]. For example, one gets

Prob (S_C=1) = 4 Prob ([1]) (9.1)

Prob (S_C=2) = 12 Prob ([2][1]) (9.2)

Prob (S_C=3) = 16 Prob ([2][2][1]) + 32 Prob ([2][1] over [2]) (9.3)

Prob (S_C=4) = 20 Prob ([2][2][2][1]) + 80 Prob ([2][2][1] over [2]) + 40 Prob ([1] over [2][3][1])

+ 40 Prob ([2][1] over [2][2]) + 32 Prob ([3][1] over [2][2]) . (9.4)

Comparing Eq. (8) with Eq. (9), we see that while the terms are the same, the coefficients of the graphs are different. In calculating Prob($s_1=r$), we sum over all FSC's of r sites such that a specified site in the FSC, and at its boundary, while in calculating Prob($s_c=r$), we sum over all FSC's in which a given *bond* is a boundary bond of an FSC. However, the number of boundary sites of an FSC is a linear function of the number of boundary bonds. Hence, for large r, we expect that

$$\text{Prob}(s_1=r)\approx K\,\text{Prob}(s_c=r)\ , \qquad (10)$$

where K is a finite constant less than 1. In two dimensions, Prob($s_c=r$) varies as $r^{-11/8}$ for large r. Hence, we get

$$\text{Prob}(s_1=r)\sim r^{-11/8} \qquad (11)$$

for large r. There is a more direct, but approximate, relationship between the properties of the conventional "direct" avalanches initiated by adding particles, and the first inverse avalanche defined here. Let n be the number of times site i topples in going from C to C_f in Eq. (1). Then it is known that all sites that topple at least j times in the avalanche form a compact cluster with no holes [6]. Let us call this cluster T_j. Then the simplest possibility consistent with Eq. (2) is that

$$T_j=F_{n+1-j} \quad \text{for } 1\le j\le n\ . \qquad (12)$$

Experimentation with small systems shows that this is usually true. (Its general validity was erroneously claimed in [2].) However, there are exceptions. One simple example is shown in Fig. 1. In our simulations, we have found that $s_1\neq s_{mt}$ in approximately 8% of the cases

(a)

2	2	4	4
3	4	4	4
4	4	4	4
4	2	4	2

(b)

4	1	4	2
2	2	2	4
4	2	3	3
2	2	3	4

FIG. 1. An example where $s_1\neq s_{mt}$ on a 4×4 lattice. A particle is added at the hatched site in a configuration shown in (a). The final configuration is shown in (b). It is easy to see that in this case the four central sites topple twice, so $s_{mt}=4$. But $s_1=5$.

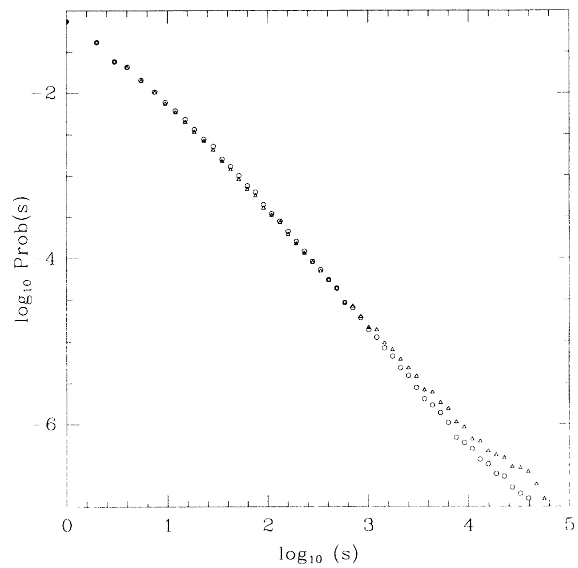

FIG. 2. Log-log plot of Prob($s_{mt}=s$) (circles) and Prob($s_1=s$) (triangles) vs s. The data are for 128 000 avalanches on a lattice of size 513×513.

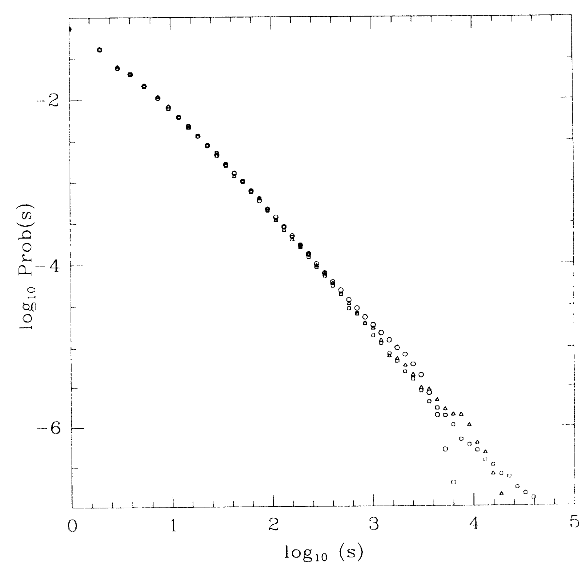

FIG. 3. Log-log plot of Prob($s_{mt}=x$) vs x for lattices of three different sizes: $L=129$ (circles), 257 (triangles), and 513 (squares). Each data point is averaged over $>10^5$ avalanches.

(here mt denotes maximum toppling; s_{mt} is the number of sites that topple the maximum number of times in a given avalanche). Even though s_1 and s_{mt} are not always equal, it is reasonable to argue that on the average they scale similarly, and hence

$$\mathrm{Prob}(s_{mt}=r)\sim r^{-11/8} \tag{13}$$

for large r.

We have verified these conclusions by numerical simulations. In Fig. 2 we show the frequency distribution of the inverse avalanche by size. To minimize boundary effects, we dropped a particle at random and each tenth particle was dropped on a small 3×3 square in the center of the square. We collected statistics for the particles dropped on the central square. The data for s_1 was binned so that the ith bin contains data for $a_i \le s_1 \le b_i$, where $a_i = b_{i-1}+1$ and b_i is the integer part of $6a_i/5$. We have plotted the averaged frequency distribution against the midpoint value of the bin interval, calculated from a data of 128 000 avalanches. On the same graph, we also show the frequency distribution of s_{mt}. It is seen that the experimental distribution for s_1 is very close to that for s_{mt}, except for very large values of s_1, where $\mathrm{Prob}(s_1)$ shows stronger finite size corrections. However, even for $r=4$, $\mathrm{Prob}(s_1=r)$ differs from $\mathrm{Prob}(s_{mt}=r)$ by a fraction of 1%. In the middle range $10\le r\le 1000$, both graphs have a linear segment, and the numerically determined slope is 1.37 ± 0.03, which is quite consistent with our theoretical prediction. In Fig. 3 we have plotted the frequency distribution for s_{mt} for lattices of three different sizes $L=129$, 257, and 513. This is based on samples of 10^6, 1.03×10^5, and 1.28×10^5 avalanches, respectively. Graphs for smaller sizes show substantial curvature, but for the largest sizes, we get the slope value quoted above. The error bars are our subjective estimate based on the goodness of fit in the scaling range.

Calculating the statistics of second, third, etc., inverse avalanches is a difficult problem, of which we have little theoretical understanding. Another question which is intriguing is formulating the precise relationship between the sets F_i and T_i, and explaining why the relationship (12) holds in a large majority of cases, but not always. We hope that future studies will throw some light on these questions.

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